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Some Partitions Associated with a Partially Ordered Set

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For any partially ordered set P, let $d_k(P)(\hat{d}_k(P))$ denote the cardinality of the largest subset of P obtained by taking the union of k antichains (chains). Then there exists a partition $\Delta = \{\Delta_1 \geqslant \Delta_2 \geqslant \cdots \geqslant \Delta_l\}$ of $|P|$ such that $d_k(P) = \Delta_1 + \Delta_2 + \cdots + \Delta_k$ and $\hat{d}_k(P) = \Delta_1^* + \Delta_2^* + \cdots + \Delta_k^*$ for each k, where Δ^* denotes the partition conjugate to Δ. This result can be used to prove a general class of "Dilworth-type" theorems for subfamilies of P.

1. INTRODUCTION

Let P be a finite partially ordered set. An *antichain* in P is a subset which contains no chains of length two. In 1950, Dilworth [4] proved that the maximum size of an antichain in P is equal to the smallest integer d such that P can be partitioned into d chains. In [7], Greene and Kleitman extended Dilworth's theorem to more general subsets of P called k-*families*. By definition, a subset $A \subseteq P$ is a k-family if A contains no chains of length $k + 1$. The central result of [7] can be described as follows.

Let $d_k(P)$ denote the maximum size of a k-family in P, and let $\mathscr{C} = \{C_1, C_2, \ldots, C_q\}$ be a partition of P into chains C_j. Define

$$\beta_k(\mathscr{C}) = \sum_{i=1}^{q} \min\{k, |C_i|\}.$$

Since each chain C_i meets every k-family at most k times, it follows that $d_k(P) \leqslant \beta_k(\mathscr{C})$. That is, every partition induces a bound on the maximum size of a k-family. Then the following is true.

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THEOREM 1.1 [7]. *For any partially ordered set P, and any positive integer k*

$$d_k(P) = \min \beta_k(\mathscr{C}),$$

where the minimum is taken over all partitions $\mathscr{C}$ of P into chains.

This reduces to Dilworth's theorem when $k = 1$. A partition $\mathscr{C}$ which minimizes $\beta_k(\mathscr{C})$ is called a *k-saturated partition of P* (into chains). In fact, a somewhat stronger result was obtained in [7]:

THEOREM 1.2 [7]. *For any $k \geq 1$, there exists a partition which is simultaneously k-saturated and $(k+1)$-saturated.*

In this paper, we consider the analogs of Theorems 1.1 and 1.2 when the concepts of "chain" and "antichain" are reversed. That is, we consider partitions of P into *antichains* (instead of chains), which induce bounds on the size of subfamilies containing no *antichains* of size $k+1$. We prove that the analogs of Theorems 1.1 and 1.2 remain valid (Theorems 1.3 and 1.4) and in the process derive a surprising relationship between the numbers $d_k(P)$ and their "complementary" counterparts (Theorem 1.6).

First some notation and terminology: If B is a subset of P which contains no antichains of size $k+1$, we call B a *k-cofamily* of P. By Dilworth's theorem, every k-cofamily can be expressed as the union of k chains. Let $\hat{d}_k(P)$ denote the size of the largest k-cofamily in P.

For each $i > 1$, define $\Delta_i(P) = d_i(P) - d_{i-1}(P)$, and $\hat{\Delta}_i(P) = \hat{d}_i(P) - \hat{d}_{i-1}(P)$. By convention, we also define $d_0(P) = \hat{d}_0(P) = 0$, so that $\Delta_1(P) = d_1(P)$ and $\hat{\Delta}_1(P) = \hat{d}_1(P)$.

If $\mathscr{A} = \{A_1, A_2, \ldots, A_r\}$ is a partition of P into antichains, define

$$\hat{\beta}_h(\mathscr{A}) = \sum_{i=1}^{r} \min\{|A_i|, h\}.$$

Clearly, $\hat{d}_h(P) \leq \hat{\beta}_h(\mathscr{A})$ for every partition $\mathscr{A}$.

We can now state the principal results. The proofs will be deferred until the next section.

THEOREM 1.3. *For any partially ordered set P, and any positive integer h,*

$$\hat{d}_h(P) = \min_{\mathscr{A}} \hat{\beta}_h(\mathscr{A}),$$

where the minimum is taken over all partitions of P into antichains.

THEOREM 1.4. *For any $h \geqslant 1$, there exists a partition of P into antichains which is h-saturated and $(h+1)$-saturated. (Here $\mathcal{O}\!\!\!\ell$ is h-saturated if $\hat{d}_h(P) = \hat{\beta}_h(\mathcal{O}\!\!\!\ell)$.)*

(It can be shown by examples that there need not always exist partitions which are simultaneously h-saturated for all h.)

THEOREM 1.5. (i) *$\Delta_1(P) \geqslant \Delta_2(P) \geqslant \cdots \geqslant \Delta_l(P)$, where l is the length of the longest chain in P.*

(ii) *$\hat{\Delta}_1(P) \geqslant \hat{\Delta}_2(P) \geqslant \cdots \geqslant \hat{\Delta}_d(P)$, where d is the size of the largest antichain in P.*

Theorem 1.5 shows that the numbers Δ_i and $\hat{\Delta}_i$ form the parts of a partition of the integer $|P|$, arranged in decreasing order. Denote these partitions by $\Delta(P)$ and $\hat{\Delta}(P)$, respectively. (Part (i) of Theorem 1.5 was proved in [7].)

THEOREM 1.6. *$\Delta(P) = \{\Delta_1(P) \geqslant \Delta_2(P) \geqslant \cdots \geqslant \Delta_l(P)\}$ and $\hat{\Delta}(P) = \{\hat{\Delta}_1(P) \geqslant \hat{\Delta}_2(P) \geqslant \cdots \geqslant \hat{\Delta}_d(P)\}$ are conjugate partitions of $|P|$. That is, $\hat{\Delta}_h(P)$ is equal to the number of parts in $\Delta(P)$ of size $\geqslant h$, for each $h = 1, 2, \ldots, d$.*

In particular, it follows from Theorem 1.6 that the $d_k(P)$'s can be determined from the $\hat{d}_k(P)$'s and vice versa.

We illustrate Theorems 1.3–1.5 by the following example. Let P be the partially ordered set shown in Fig. 1. It is easy to see that $d_1(P) = 4$ and

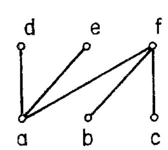

FIGURE 1

$d_2(P) = 6$, so that $\Delta(P) = \{4, 2\}$. According to Theorem 1.6, $\hat{\Delta}(P)$ is the partition $\{2, 2, 1, 1\}$, which means $\hat{d}_1(P) = 2$, $\hat{d}_2(P) = 4$, $\hat{d}_3(P) = 5$, and $\hat{d}_4(P) = 6$. To see that these numbers are correct, we construct a table of h-cofamilies and h-saturated partitions of P into antichains, for $h = 1, 2$, and 3 ($\hat{d}_4(P) = 6$ holds trivially):

$h =$	h-cofamily	h-saturated partition
1	$\{a, d\}$	$\{a, b, c\} \cup \{d, e, f\}$
2	$\{a, d, c, f\}$	$\{a, b, c\} \cup \{d, e, f\}$
3	$\{a, b, c, d, f\}$	$\{a\} \cup \{f\} \cup \{b, c, d, e\}$

Note also that $\{a, b, c\} \cup \{d, e, f\}$ is 1-saturated and 2-saturated, while $\{a\} \cup \{f\} \cup \{b, c, d, e\}$ is 2-saturated and 3-saturated. It can be checked easily that *no* partition is simultaneously 1-, 2-, and 3-saturated.

Before proceeding to the proofs, we make several additional remarks concerning the background and motivation for these problems.

(1) It is trivial and well known that Dilworth's theorem remains true when the roles of chain and antichain are reversed. This fact suggested looking for complementary analogs of the results in [7]. To prove Theorem 1.3 when $h = 1$ (i.e., to find a 1-saturated partition of P into antichains) let A_i denote the set of elements of height i in P, for $i = 1, 2, \ldots, l$. (The *height* of an element $x \in P$ is the length of the longest chain whose top is x.) Then $\mathcal{A} = \{A_1, A_2, \ldots, A_l\}$ is a 1-saturated partition.

(2) The theory of *perfect graphs*, developed by Berge (see [2]), Fulkerson [6], and Lovasz [9], suggests that Dilworth-type theorems tend to come in complementary pairs, under very general conditions. If G is any undirected graph, we can interpret "Dilworth's theorem" as the following statement about G (which may or may not be true). *The vertices of G can be covered by d totally related (complete) subgraphs, if d is the size of the largest totally unrelated (independent) set in G.* By definition, a graph G is *perfect* if Dilworth's theorem holds for every subgraph of G. The *perfect graph theorem* states that, if G is perfect, then so is its complement G^* (the graph whose edges are the "nonedges" of G). One can examine various analogs of Theorems 1.1–1.5 in the more general context of graph theory and we will do so in Section 4. Surprisingly, Theorem 1.1 does not imply Theorem 1.3 in general, but the stronger Theorem 1.2 implies Theorem 1.4.

(3) Many of the results in this paper were motivated by an important class of examples arising from the theory of permutations. If

$$\sigma = \langle a_1, a_2, \ldots, a_n \rangle$$

is a sequence of distinct integers, we can associate with σ a partially ordered set P_σ whose chains and antichains correspond to increasing and decreasing subsequences of σ. Define P_σ to be the set of all pairs (a_i, i), with the usual product ordering. In [8] it was shown that $\Delta(P_\sigma)$ is identical to the partition associated with σ by a procedure known as Schensted's algorithm (see [2]). This algorithm constructs a certain Young tableau based on the elements of σ, and $\Delta(P_\sigma)$ turns out to be its shape. When the order of σ is reversed, it is known that the "Schensted tableau" is transformed into its transpose. This proves Theorem 1.6 directly when P is of the form P_σ, since reversing the order of σ has the effect of interchanging chains and antichains.

It should be noted that when P is not of this form the situation is much more complicated: In general, there need not exist a partially ordered set $\hat{P}$ whose chains are the antichains of P and vice versa. In fact, it can be shown that this occurs *only* when $P \approx P_\sigma$ for some σ [5].

2. Proofs of the Main Results

To prove Theorems 1.3–1.6, we will need to use Theorem 1.2 and Theorem 1.5(i) (proved in [7]), and also one additional result from [7], not mentioned in the Introduction:

Theorem 2.1 [7, Theorem 3.10]. *Let k be such that $\Delta_k(P) > \Delta_{k+1}(P)$. Then there exists an element $x \in P$ which is contained in every k-family and $(k+1)$-family of maximum size.*

In [7], this result was used to prove Theorem 1.2. On the other hand, it can be derived trivially from Theorem 1.2 by the following argument. Let $\mathscr{C}$ be a partition of P into chains which is both k-saturated and $(k+1)$-saturated. There must be at least one chain of length $\leqslant k$, since otherwise $\Delta_k(P) = \Delta_{k+1}(P)$. Choose x to be any element of such a chain, and x has the desired property.

Except for the use of Theorems 1.2, 1.5(i), and 2.1, the arguments in this section are self-contained.

It is sometimes useful to observe that a k-saturated partition remains k-saturated if all of the chains of length $<k$ are broken up into singletons. Thus a collection of chains $C_1, C_2, \ldots, C_h$ of length $\geqslant k$ (together with the set $S = P - \bigcup_1^h C_i$ of singletons) determines a k-saturated partition of P if and only if $d_k(P) = kh + |S|$. For convenience of notation, we write $\mathscr{C} = \{C_1, C_2, \ldots, C_h; S\}$ when $\mathscr{C}$ is obtained in this way. Similarly, if $A_1, A_2, \ldots, A_k$ is a collection of antichains and $T = P - \bigcup_1^k A_i$, define a partition $\mathscr{A} = \{A_1, A_2, \ldots, A_k; T\}$ similarly.

The strategy will be as follows. We first prove Theorem 1.6, which allows the $\hat{d}_h(P)$'s to be expressed in terms of the $d_k(P)$'s, and also verifies Theorem 1.5(ii). Using this information, we may then easily prove Theorem 1.4 (and hence, Theorem 1.3).

Let $\Delta^*(P) = \{\Delta_1^*(P) \geqslant \Delta_2^*(P) \geqslant \cdots \geqslant \Delta_d^*(P)\}$ denote the partition of $|P|$ conjugate to $\Delta(P)$.

Lemma 2.2. *For all $h \geqslant 1$, $\hat{d}_h(P) \leqslant \Delta_1^*(P) + \Delta_2^*(P) + \cdots + \Delta_h^*(P)$.*

Proof. Let $C = C_1 \cup \cdots \cup C_h$ be an h-cofamily of P, and let $S = P - C$. Define $k = \Delta_h^*(P)$, and $\mathscr{C} = \{C_1, C_2, \ldots, C_h; S\}$. Then

$d_k(P) \leqslant \beta_k(\mathscr{C}) \leqslant hk + |S| = hk + |P| - |C|$. Hence, $|C| \leqslant |P| - d_k(P) + hk = |P| - (\Delta_1(P) + \Delta_2(P) + \cdots + \Delta_k(P)) + hk = hk + \Delta_{k+1}(P) + \cdots + \Delta_l(P)$. But the last expression is equal to $\Delta_1{}^*(P) + \Delta_2{}^*(P) + \cdots + \Delta_h{}^*(P)$. (This can be seen easily by looking at the Ferrer's diagram of $\Delta(P)$.)

LEMMA 2.3. *Let $\mathscr{C} = \{C_1, C_2, ..., C_h; S\}$ be a k-saturated partition of P (into chains), with $|C_i| \geqslant k$ for each i. Define $C = C_1 \cup C_2 \cup \cdots \cup C_h$. Then*

(i) $\Delta_k(P) \geqslant h \geqslant \Delta_{k+1}(P)$,

(ii) *C is an h-cofamily of maximum size,*

(iii) $|C| = \hat{d}_h(P) = \Delta_1{}^*(P) + \Delta_2{}^*(P) + \cdots + \Delta_h{}^*(P)$.

Proof. Statement (i) follows trivially from the inequalities

$$d_{k-1}(P) \leqslant (k-1)h + |S|,$$
$$d_k(P) = kh + |S|,$$
$$d_{k+1}(P) \leqslant (k+1)h + |S|.$$

To prove (ii), suppose that $C' = C_1' \cup C_2' \cup \cdots \cup C_h'$ is an h-cofamily with $|C'| > |C|$. If $S' = P - C'$, then $|S'| < |S|$. Hence if

$$\mathscr{C}' = \{C_1', C_2', ..., C_h'; S'\},$$

then $\beta_k(\mathscr{C}') \leqslant kh + |S'| < kh + |S| = \beta_k(\mathscr{C})$, which is impossible, since $\mathscr{C}$ is k-saturated. To prove (iii), observe that $|S| = d_k(P) - kh$, and hence $|C| = |P| - |S| = |P| - (\Delta_1(P) + \cdots + \Delta_k(P)) + kh = \Delta_{k+1}(P) + \cdots + \Delta_l(P) + kh$. But it follows easily from (i) that

$$\Delta_{k+1}(P) + \cdots + \Delta_l(P) + kh = \Delta_1{}^*(P) + \Delta_2{}^*(P) + \cdots + \Delta_h{}^*(P),$$

as desired.

COROLLARY 2.4. $\hat{d}_h(P) = \Delta_1{}^*(P) + \Delta_2{}^*(P) + \cdots + \Delta_h{}^*(P)$ *whenever* $h = \Delta_k(P)$ *for some k.*

Proof. By Lemma 2.3, $\hat{d}_h(P) = \Delta_1{}^*(P) + \Delta_2{}^*(P) + \cdots + \Delta_h{}^*(P)$ whenever h corresponds to the number of chains of length $\geqslant k$ in some k-saturated partition $\mathscr{C}$. Let k be given. Then, by Theorem 1.2, there exists a partition

$$\mathscr{C} = \{C_1, C_2, ..., C_h; S\}$$

which is *both* k-saturated and $(k-1)$-saturated (and such that $|C_i| \geqslant k$ for each i). In this case Lemma 2.3(i) implies $h = \Delta_k(P)$, and the corollary follows.

The next step is to remove the above restriction on h and obtain a proof of Theorem 1.6 in all cases.

Proof of Theorem 1.6. We wish to prove

$$\hat{d}_h(P) = \Delta_1{}^*(P) + \cdots + \Delta_h{}^*(P) \text{ for all } h \geqslant 1.$$

Assume inductively that this holds for all partially ordered sets of size $|P| - 1$. If $h = \Delta_k(P)$ for some k, we are done by Corollary 2.4. Hence, we can assume that $\Delta_k(P) > h > \Delta_{k+1}(P)$ for some $k \geqslant 1$. (Note that if $h > \Delta_1(P)$ the result is trivial, since P is an h-cofamily and $\hat{d}_h(P) = \Delta_1{}^*(P) + \cdots + \Delta_h{}^*(P) = |P|$.) By Theorem 2.1, there exists an element $x \in P$ such that x is contained in every k-family of size $d_k(P)$, and also every $(k+1)$-family of size $d_{k+1}(P)$. Let $P' = P - x$. Then $d_k(P') = d_k(P) - 1$ and $d_{k+1}(P') = d_{k+1}(P) - 1$, so that $\Delta_{k+1}(P') = \Delta_{k+1}(P)$. Furthermore, $\Delta_k(P')$ is equal to either $\Delta_k(P)$ or $\Delta_k(P) - 1$.

We claim that

$$\Delta_1{}^*(P) + \Delta_2{}^*(P) + \cdots + \Delta_h{}^*(P) = \Delta_1{}^*(P') + \Delta_2{}^*(P') + \cdots + \Delta_h{}^*(P').$$

To see this, observe that since $h > \Delta_{k+1}(P) = \Delta_{k+1}(P')$ and $\Delta_k(P') \geqslant \Delta_k(P) - 1 \geqslant h$ we have $\Delta_i(P) \geqslant h$ if and only if $\Delta_i(P') \geqslant h$. Hence,

$$\begin{aligned}
\Delta_1{}^*(P) + \cdots + \Delta_h{}^*(P) &= \sum_{i=1}^{l} \min\{h, \Delta_i(P)\} \\
&= \sum_{i=1}^{l} \min\{h, \Delta_i(P')\} \\
&= \Delta_1{}^*(P') + \cdots + \Delta_h{}^*(P').
\end{aligned}$$

By the inductive hypothesis, we can find an h-cofamily in P' (and hence in P) of size $\Delta_1{}^*(P) + \cdots + \Delta_h{}^*(P)$. Hence, $\hat{d}_h(P) \geqslant \Delta_1{}^*(P) + \cdots + \Delta_h{}^*(P)$. But Lemma 2.2 shows that equality must occur, and the proof of Theorem 1.6 is complete.

Now that we can describe $\hat{\Delta}(P)$ in terms of $\Delta(P)$, it is easy to show that h- and $(h+1)$-saturated partitions of P into antichains always exist (i.e., to prove Theorem 1.4).

Proof of Theorem 1.4. Let h be given, and define $k = \hat{\Delta}_{h+1}(P)$. Let $A = A_1 \cup A_2 \cup \cdots \cup A_k$ be any k-family of maximum size, and define $\mathcal{C} = \{A_1, A_2, \ldots, A_k; T\}$, where $T = P - A$. Since $|A| = d_k(P) = \Delta_1(P) + \cdots + \Delta_k(P)$, we have $|T| = \Delta_{k+1}(P) + \cdots + \Delta_l(P)$. This implies $\beta_h(\mathcal{C}) \leqslant kh + |T| = kh + \Delta_{k+1}(P) + \cdots + \Delta_l(P) = \hat{\Delta}_1(P) + \hat{\Delta}_2(P) + \cdots + \hat{\Delta}_h(P) = \hat{d}_h(P)$ by Theorem 1.6. Hence $\mathcal{C}$ is h-saturated. Similarly,

$\hat{\beta}_{h+1}(\mathcal{A}) \leqslant k(h+1) + |T| = \hat{d}_h(P) + \hat{\Delta}_{h+1}(P) = \hat{d}_{h+1}(P)$. Hence, $\mathcal{A}$ is both h-saturated and $(h+1)$-saturated, as desired.

The results proved in this section indicate a very close relationship between i-families and j-saturated "complementary" partitions, and between j-cofamilies and i-saturated partitions, for certain values of i and j. In a sense, j-cofamilies *are* i-saturated partitions, provided that i and j are related properly. We make this connection precise in the following theorem.

THEOREM 2.5. (i) *Let $C = C_1 \cup C_2 \cup \cdots \cup C_h$ be an h-cofamily of size $\hat{d}_h(P)$, and let k be such that $\hat{\Delta}_h(P) \geqslant k \geqslant \hat{\Delta}_{h+1}(P)$. Then $|C_i| \geqslant k$ for each i and $\mathscr{C} = \{C_1, C_2, \ldots, C_h; P - C\}$ is a k-saturated partition.*

(ii) *Let $\mathscr{C} = \{C_1, C_2, \ldots, C_h; T\}$ be a k-saturated partition, with $|C_i| \geqslant k$ for each i. Then $\hat{\Delta}_h(P) \geqslant k \geqslant \hat{\Delta}_{h+1}(P)$ and $C = C_1 \cup C_2 \cup \cdots \cup C_h$ is an h-cofamily of size $\hat{d}_h(P)$.*

A similar relationship holds between k-families and h-saturated partitions into antichains. We remark that the condition $\hat{\Delta}_h(P) \geqslant k \geqslant \hat{\Delta}_{h+1}(P)$ is equivalent to the condition $\Delta_k(P) \geqslant h \geqslant \Delta_{k+1}(P)$, as a glance at the Ferrers diagram of $\Delta(P)$ shows.

Proof. Part (ii) follows from Lemma 2.3 and the above remark. To prove part (i), observe that $\beta_k(\mathscr{C}) \leqslant kh + |P - C| = kh + \hat{\Delta}_{h+1}(P) + \cdots + \hat{\Delta}_d(P) = d_k(P)$, since $\hat{\Delta}_h(P) \geqslant k \geqslant \hat{\Delta}_{h+1}(P)$. This proves that $\mathscr{C}$ is k-saturated. Trivially, $|C_i| \geqslant \hat{\Delta}_h(P)$ for all i, which implies $|C_i| \geqslant k$.

Once it is known that h-saturated partitions exist, it is possible to derive a number of additional properties of h-cofamilies. We mention several corollaries, which are counterparts of results obtained for k-families by Greene and Kleitman in [7].

THEOREM 2.6. *If $\hat{\Delta}_1(P) > \hat{\Delta}_{h+1}(P)$, then there exists an element $x \in P$ which is contained in every h-cofamily and $(h+1)$-cofamily of maximum size.*

Proof. Let $\mathcal{A} = \{A_1, \ldots, A_k; T\}$ be a partition of P into antichains which is h-saturated and $(h+1)$-saturated (with each $|A_i| \geqslant h$). Since $\Delta_1{}^*(P) > \Delta_{h+1}^*(P)$, T must be nonempty. Choose x to be any member of T.

THEOREM 2.7. *Let $\mathscr{F}_h(P)$ denote the set of all h-cofamilies of maximum size in P. If every set of $(h+1)$ members of $\mathscr{F}_h(P)$ has nonempty intersection, there is an element $x \in P$ which is common to all members of $\mathscr{F}_h(P)$.*

Proof. By the previous result, $\mathscr{F}_h(P)$ fails to have nonempty intersection only if $\Delta_1^*(P) = \Delta_{h+1}^*(P)$. However, in this case there exist $h + 1$ pairwise disjoint chains $C_1, C_2, \ldots, C_{h+1}$ of maximum length $(\hat{\Delta}_1(P))$ in P. By removing the chains one at a time, we obtain $h + 1$ different maximum-sized h-cofamilies without a common member.

The last theorem (proved for k-families in [7]) bears a formal resemblance to Helly's theorem for convex regions in h-dimensional Euclidean space (see [3]). However, the h-cofamilies of P do not really have the "Helly property," since the statement of Theorem 2.7 need not hold for arbitrary subcollections of $\mathscr{F}_k(P)$.

In the special case $h = 1$, Theorem 2.7 can be restated as follows. *If any two maximum-length chains of P have a common member, then all such chains have a common member.* The reader may find it an amusing exercise to construct a direct proof of this statement.

3. Perfect Graph Theorems

We conclude with some remarks about the relationship between the results in this paper and the theory of perfect graphs. (Recall the definition of perfect graphs in Section 1.) The first results are negative.

Let G be the graph illustrated in Fig. 2, and let G^* denote its complement (shown in Fig. 3). It is well known that both G and G^* are perfect

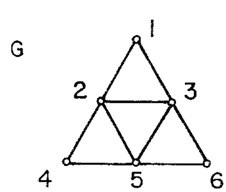

Figure 2

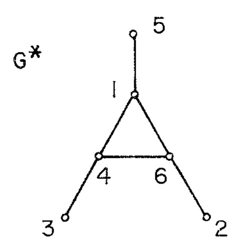

Figure 3

graphs, but neither represents the relation of comparability in a partially ordered set (see [3]). That is, it is not possible to assign a transitive orientation to the edges of G or G^*. On the other hand, all of the proper subgraphs of G and G^* have this property. With the obvious extension of our previous notation, we compute

$$d_1(G) = 3, \qquad \hat{d}_1(G) = d_1(G^*) = 3;$$
$$d_2(G) = 4, \qquad \hat{d}_2(G) = d_2(G^*) = 5;$$
$$d_3(G) = 6, \qquad \hat{d}_3(G) = d_3(G^*) = 6.$$

From G and G^*, the following conclusions can be drawn.

(1) *Theorem* 1.5 *need not hold for perfect graphs. That is, it need not be true that* $\Delta_1 \geqslant \Delta_2 \geqslant \cdots \geqslant \Delta_l$. (This is illustrated by G.)

(2) *If Theorem* 1.5 *holds for all subgraphs of a graph, it need not hold for its complement.* (This is illustrated by G^*.)

(3) *Theorem* 1.1 (*and hence Theorem* 1.2) *need not hold for perfect graphs. That is, k-saturated partitions* (*into complete subgraphs*) *do not always exist, for arbitrary perfect graphs.* (This is illustrated by G when $k = 2$. For a partition to be 2-saturated, it would have to consist of two parts, or one part and two singletons, or four singletons. None of these possibilities exists.)

(4) *If Theorem* 1.1 *holds for all subgraphs of a graph, it need not hold for its complement.* (This is illustrated by G^*. The partition $\mathscr{C} = \{15, 34, 26\}$ is 1-saturated, and the partition $\mathscr{C}' = \{146, 2, 3, 5\}$ is 2-saturated and 3-saturated. Thus, Theorem 1.1 holds in G^* (and all of its subgraphs). Yet, as we have already observed, it fails to hold in G.)

(5) *Theorem* 1.1 *need not imply Theorem* 1.2. (This is illustrated by G^*. We have already shown that k-saturated partitions exist for $k = 1, 2, 3$. However, it is impossible to find a partition which is both 1-saturated and 2-saturated.)

The last two observations leave open a small possibility for obtaining a "perfect" theorem for k-families, which turns out to be true:

THEOREM 3.1. *Let G be a graph such that every subgraph satisfies the conclusion of Theorem* 1.2 (*That is, k- and* $(k+1)$-*saturated partitions into complete subgraphs exist for every value of k.*) *Then G^* has the same property.*

We omit the proof of Theorem 3.1, and merely mention that Theorem 1.2 alone implies all of the lemmas and corollaries used in this paper (in particular, Theorem 1.5(i) and Lemma 2.1).

We do not know the relationship between the class of graphs described in Theorem 3.1 and the other more familiar classes of perfect graphs (see [2]).

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